

THE COOLIDGE-NAGATA CONJECTURE HOLDS FOR CURVES WITH MORE THAN FOUR CUSPS

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ABSTRACT. Let $E \subseteq \mathbb{P}^2$ be a rational curve defined over complex numbers which has only locally irreducible singularities. The Coolidge-Nagata conjecture states that E is *rectifiable*, i.e. it can be transformed into a line by a birational automorphism of \mathbb{P}^2 . We show that if it is not rectifiable then the tree of the exceptional divisor for its minimal embedded resolution of singularities has at most nine maximal twigs. This settles the conjecture in case E has more than four singular points.

1. MAIN RESULT

All varieties considered are complex algebraic. An irreducible curve is *cuspidal* if and only if all its singular points are cusps, i.e. they are locally irreducible. We are interested in rational cuspidal curves embedded into the projective plane \mathbb{P}^2 . Let $\bar{E} \subseteq \mathbb{P}^2$ be such a curve. There are many examples with \bar{E} having one, two or three cusps (already among quartics). Up to a choice of coordinates on the plane there is only one known example with four cusps. It is of degree five and has parametrization $(t^3 - 1, t^5 + 2t^2, t)$. No examples with more than four cusps are known and it is expected that they do not exist. It is also expected that any rational plane cuspidal curve is *rectifiable*, i.e. there exists a birational automorphism of the plane, such that the proper transform of the curve is a line. This is known as the *Coolidge-Nagata conjecture/problem*. The conjecture has been verified for all known examples. We show that even if rational plane cuspidal curves with more than four cusps do exist, the Coolidge-Nagata conjecture necessarily holds for them.

Theorem 1.1. *Let $\bar{E} \subseteq \mathbb{P}^2$ be a rational cuspidal curve defined over complex numbers. If \bar{E} has more than four cusps then there exists a birational automorphism of \mathbb{P}^2 which transforms \bar{E} into a line.*

We show in fact that if $\bar{E} \subseteq \mathbb{P}^2$ is non-rectifiable then the tree of the exceptional divisor for its minimal embedded resolution has at most nine maximal twigs (cf. 5.5).

The structure of the paper is as follows. In section 3 we prove two main inequalities. In section 4 we complete the proof of the theorem by dealing with the case of five cusps and ten maximal twigs. In section 5 we exclude the case of four cusps and ten maximal twigs.

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2. PRELIMINARIES

2.1. Open surfaces. We recall some results from the theory of non-complete surfaces, also to settle the notation. For a complete treatment the reader is referred to [Miy01].

Let T be a nonzero reduced simple normal crossing divisor on a smooth complete surface X . We denote the Iitaka-Kodaira dimension of T by $\kappa(T)$. If R is a reduced divisor with support contained in T we define $\beta_T(R) = R \cdot (T - R)$ and call it a *branching number of R in T* . If R is irreducible we say that R is a *tip* or a *branching component* if $\beta_T(R) = 1$ or $\beta_T(R) \geq 3$ respectively.

The *arithmetic genus of T* is $p_a(T) = \frac{1}{2}T \cdot (K + T)$, where K is the canonical divisor (class) on X . T is a *rational tree* if all its components are rational and the dual graph of T contains no loops. In this case $p_a(T) = 0$. We call T a *chain* if it has no branching components. If $T = T_1 + \dots + T_k$, is a decomposition of a rational chain into irreducible components, such that $T_i \cdot T_{i+1} = 1$ for $i < k$, then we write $T = [-T_1^2, -T_2^2, \dots, -T_k^2]$. By $(m)_p$ we mean a sequence (m, m, \dots, m) of length p . An (n) -curve is a smooth rational curve with self-intersection n .

We define the *discriminant of T* by $d(T) = \det(-Q(T))$, where $Q(T) = (T_i \cdot T_j)_{i,j \leq k}$ is the intersection matrix of T . We put $d(0) = 1$. The following formula follows from elementary properties of determinants (cf. [Rus80]).

Lemma 2.1. *Let S and T be reduced simple normal crossing divisors, such that $S \cdot T = 1$ and let $S_0 \subseteq S$ and $T_0 \subseteq T$ be the irreducible components for which $S_0 \cdot T_0 = 1$. Then*

$$d(S + T) = d(S)d(T) - d(S - S_0)d(T - T_0).$$

In particular, if $S = S_0$ then $d(T + S_0) = -S_0^2 d(T) - d(T - T_0)$.

Lemma 2.2. *Let T be a rational chain which contains no (-1) -curves and has a negative definite intersection matrix.*

- (i) *If $d(T) = 2$ then $T = [2]$.*
- (ii) *If $d(T) = 3$ then $T = [2, 2]$ or $T = [3]$.*
- (iii) *If $d(T) = 4$ then $T = [2, 2, 2]$ or $T = [4]$.*
- (iv) *If $d(T) = 5$ then $T = [2, 2, 2, 2]$ or $T = [2, 3]$ or $T = [3, 2]$ or $T = [5]$.*
- (v) *If $d(T) = 6$ then $T = [2, 2, 2, 2, 2]$ or $T = [6]$.*

Proof. We just note that if T_1 is the tip of T and T_2 is the component meeting T_1 then by 2.1 $d(T) = -T_1^2 d(T - T_1) - d(T - T_1 - T_2) > d(T - T_1)$, so all chains T with given discriminant can be found by induction. \square

Assume now that T is a rational tree without non-branching (-1) -curves and with intersection matrix which is not negative definite. Assume also that T is not a chain and that the intersection matrices of all its maximal twigs are negative definite. Let $T_i = T_{i,1} + \dots + T_{i,k_i}$, where $T_{i,1}$ assumed to be a tip of T , $i = 1, \dots, t$, be all its maximal twigs. We put $e(T_i) = d(T_i - T_{i,1})/d(T_i)$ and $\delta(T_i) = 1/d(T_i)$. We define

$$\delta(T) = \sum_{i=1}^s \delta(T_i) \quad \text{and} \quad e(T) = \sum_{i=1}^s e(T_i).$$

Assume that T is as above and $\kappa(K + T) \geq 0$. We have the Zariski decomposition $K + T = (K + T)^+ + (K + T)^-$, where $(K + T)^+$ is numerically effective and $(K + T)^-$ is effective, either empty or having a negative definite intersection matrix. Moreover, $(K + T)^+ \cdot B = 0$ for any curve B contained in $\text{Supp}(K + T)^-$. We define $\text{Bk } T$, the *bark* of T , as a unique \mathbb{Q} -divisor with support contained in the sum of maximal twigs of T and satisfying

$$\text{Bk } T \cdot T_0 = \beta_T(T_0) - 2$$

for every component T_0 of every maximal twig of T .

Lemma 2.3. *Let T be a rational tree as above. Let T_i be a maximal twig of T and let T_0 be a component of T_i . Denote the coefficient of T_0 in the decomposition of $\text{Bk } T$ into irreducible components by t_0 .*

- (i) *The coefficient t_0 satisfies $0 < t_0 < 1$.*
- (ii) *If T_0 meets $T - T_i$ then $t_0 = \delta(T_i)$.*
- (iii) *$(\text{Bk } T)^2 = -e(T)$.*
- (iv) *If there is no (-1) -curve A on X , for which $T \cdot A \leq 1$, then $(K + T)^- = \text{Bk } T$.*

Proof. Write $T_i = T_{i,1} + T_{i,2} + \dots + T_{i,k_i}$, where $T_{i,k}$ are irreducible and $T_{i,j} \cdot T_{i,j+1} = 1$ for $j < k_i$. Then by [Miy01, 2.3.3.4] the coefficient of $T_{i,j}$ in $\text{Bk } T$ equals $d(T_{i,j+1} + \dots + T_{i,k_i})/d(T_i)$. This gives (i), (ii) and (iii). Part (iv) follows from 2.3.11 loc. cit. \square

2.2. Rectifiability. Let $\bar{E} \subseteq \mathbb{P}^2$ be a rational curve with singular points q_1, \dots, q_c , $c > 0$. Let $\pi: X \rightarrow \mathbb{P}^2$ be a minimal embedded good resolution of singularities for \bar{E} , i.e. $\pi^* \bar{E}$ is a simple normal crossing divisor, whose all (-1) -curves are branching. Let K be the canonical divisor (class) on X and let D be the reduced total transform of \bar{E} . We denote the proper transform of \bar{E} on X by E . Write $\pi^{-1}(q_i) = Q_i$, where Q_i is reduced effective. By definition Q_i is a rational tree with negative definite intersection matrix. Since Q_i contracts to a smooth point on a surface, $d(Q_i) = d([1]) = 1$. Note that if \bar{E} is cuspidal then D is a rational tree.

Proposition 2.4. *Let \bar{E} and E be as above.*

- (i) *$\bar{E} \subseteq \mathbb{P}^2$ is rectifiable if and only if $\kappa(K + E) = -\infty$.*
- (ii) *If $\kappa(K + E) \geq 0$ then $h^0(2K + E) \geq 1$. The inequality is strict if $\kappa(K + E) = 2$.*

Proof. (i) is a theorem of Coolidge [Coo59], see also [MKM83, 2.6]. (ii), crucial for us, is [MKM83, 2.4, 3.2]. Let us recall the proof of the first part of (ii), rearranging the arguments

a bit. First of all, we can assume without loss of generality that there is no (-1) -curve on X disjoint from E . Let $m > 0$ be minimal such that $h^0(mE + m'K) \neq 0$ for some $m' \geq m$. Write

$$mE + m'K = \sum_i U_i,$$

where U_i 's are irreducible. By the choice of m , $U_i \neq E$. Note that if $(K + E) \cdot U_{i_0} < 0$ for some i_0 then $K \cdot U_{i_0} < 0$, so

$$\left(\sum_i U_i\right) \cdot U_{i_0} = m(K + E) \cdot U_{i_0} + (m' - m)K \cdot U_{i_0} < 0.$$

But in this case we would get $U_{i_0}^2 < 0$, so U_{i_0} would be a (-1) -curve disjoint from E , which contradicts the assumption. We get

$$0 \leq (K + E) \cdot (mE + m'K) = -2m + m'K \cdot (K + E),$$

so $K \cdot (K + E) > 0$. Since the numerical class of $K + E$ is nonzero, by the Riemann-Roch theorem $h^0(2K + E) \geq K \cdot (K + E) > 0$. \square

Remark 2.5. It follows from 2.4(ii) that if $\kappa(K + E) \geq 0$ then $E^2 \leq -4$ and $\deg \bar{E} \geq 6$ (and the inequality is strict if $\kappa(K + E) = 2$). The see the former note that since $\kappa(K) = -\infty$, E is not in the fixed part of $|2K + E|$, so $0 \leq E \cdot (2K + E) = -4 - E^2$. The latter follows from the equality $\pi_*(2K + E) = 2K_{\mathbb{P}^2} + \bar{E}$.

Note also that if $\bar{E} \subseteq \mathbb{P}^2$ is a general rational curve of degree d then its singularities are ordinary double points (nodes), so we compute easily $2K + E \sim (d - 6)H$, where H is a pullback of a line on \mathbb{P}^2 . Thus \bar{E} is not rectifiable for $d \geq 6$. A general rational sextic with ten nodes is an example of lowest degree.

Lemma 2.6. *Let $\bar{E} \subseteq \mathbb{P}^2$ be rational and cuspidal.*

- (i) *If $c \geq 2$ then $h^0(2(K + D)) \neq 0$.*
- (ii) *If $c \geq 3$ then $\kappa(K + D) = 2$.*
- (iii) *If $\kappa(K + D) = 2$ then $(K + D)^- = \text{Bk } D$.*

Part (ii) is explicitly stated in [Wak78] and part (i) follows from a proof there. For (iii) note that if $\kappa(K + D) = 2$ then by the Lefschetz duality $X - D$ is a smooth \mathbb{Q} -acyclic surface of general type, so it does not contain topologically contractible lines by [MT92]. Since $X - D$ is affine, it follows that there is no (-1) -curve A on X for which $A \cdot D \leq 1$, so $(K + D)^- = \text{Bk } D$ by 2.3(iv).

3. TWO INEQUALITIES

From now on we assume that $\bar{E} \subseteq \mathbb{P}^2$ is a rational cuspidal curve with cusps q_1, \dots, q_c , $c > 0$. In this case the divisor Q_i can be seen as produced by a *connected sequence of blow-ups*, i.e. we can decompose the morphism contracting Q_i to a point into a sequence of blow-ups $\sigma_1 \circ \dots \circ \sigma_s$, so that then the center of σ_{i+1} belongs to the exceptional component of σ_i for $i \geq 1$. Let C_i be the unique (-1) -curve in Q_i . Clearly, $E \cdot Q_i = E \cdot C_i = 1$. Since π is minimal, C_i is not a tip of D , so $Q_i - C_i$ has two connected components. One of them is a rational chain and the other a rational tree, C meets them in tips. We denote the maximal twigs of D by T_1, \dots, T_t .

The surface $X - D$ is \mathbb{Q} -acyclic, i.e. $H_i(X - D, \mathbb{Q}) = 0$ for $i > 0$. The group $(\text{Pic } \mathbb{P}^2) \otimes \mathbb{Q}$ is generated by \bar{E} , so $(\text{Pic } X) \otimes \mathbb{Q}$ is generated freely by the components of D . Since $D - E$ has a negative definite intersection matrix, the intersection matrix of D is not negative definite by the Hodge index theorem. If $\kappa(K + D) \geq 0$ we put $\mathcal{P} = (K + D)^+$. In the inequalities below the case $c = 1$ is somewhat special. It is convenient to introduce $\epsilon(c)$ defined to be 0 for $c > 1$ and 1 for $c = 1$.

Proposition 3.1. *If $\kappa(K + E) \geq 0$ then the following inequality holds:*

$$(\star) \quad t - \frac{1}{2}(c + \epsilon(c)) \leq \delta(D) + 1 + \mathcal{P}^2 \leq \delta(D) + 4.$$

Proof. We have $\kappa(K + D) \geq \kappa(K + E) \geq 0$. The divisor $R = D - T_1 - \dots - T_t$ is a reduced rational tree, so $p_a(R) = 0$. The rational twigs are contained in $\text{Supp Bk } D$, so they intersect \mathcal{P} trivially. We get

$$\mathcal{P} \cdot D = \mathcal{P} \cdot R = (K + D - \text{Bk } D) \cdot R = (K + R) \cdot R + \beta_D(R) - \text{Bk } D \cdot R.$$

By 2.3(ii) the coefficient in $\text{Bk } D$ of the component of the twig T_i which intersects R is $\delta(T_i)$. Thus $\mathcal{P} \cdot D = -2 + t - \delta(D)$. We have

$$\mathcal{P} \cdot E = (K + D - \text{Bk } D) \cdot E = (K + E) \cdot E + \beta_D(E) - \text{Bk } D \cdot E = -2 + c - \text{Bk } D \cdot E.$$

If $c = 1$ then E is a maximal twig of D , which gives $\mathcal{P} \cdot E = c - 1 = 0$. If $c \geq 2$ then E is disjoint from $\text{Bk } D$, hence $\mathcal{P} \cdot E = c - 2$. The formula which holds for both cases is therefore $\mathcal{P} \cdot E = c + \epsilon(c) - 2$. From 2.4(ii) we see that

$$0 \leq \mathcal{P} \cdot (2K + E) = 2\mathcal{P}(K + D) - 2\mathcal{P} \cdot D + \mathcal{P} \cdot E = 2\mathcal{P}^2 - 2\mathcal{P} \cdot D + \mathcal{P} \cdot E.$$

Thus

$$\mathcal{P}^2 \geq \mathcal{P} \cdot D - \frac{1}{2}\mathcal{P} \cdot E = t - 2 - \delta(D) - \frac{1}{2}(c + \epsilon(c) - 2) = t - 1 - \delta(D) - \frac{1}{2}(c + \epsilon(c)).$$

Finally, from the logarithmic Bogomolov-Miyaoka-Yau inequality (see [Lan03], cf. [Pal11, 2.5]) we have $\mathcal{P}^2 \leq 3\chi(X - D) = 3$, where χ denotes the Euler characteristic. \square

Proposition 3.2. *If $\kappa(K + D) = 2$ then*

$$(\diamond) \quad h^0(2K + D) + e(D) = \mathcal{P}^2 + 2 \leq 5.$$

Proof. We have $K \cdot (K + D) - 2 = (K + D)^2 = \mathcal{P}^2 + \text{Bk}^2 D = \mathcal{P}^2 - e(D)$. Since the numerical class of $K + D$ is nonzero, the Riemann-Roch theorem gives $h^0(2K + D) - h^1(2K + D) = K \cdot (K + D)$. By the logarithmic Bogomolov-Miyaoka-Yau inequality we have $\mathcal{P}^2 \leq 3$, so we see that

$$h^0(2K + D) - h^1(2K + D) + e(D) = \mathcal{P}^2 + 2 \leq 5.$$

Note that by 2.3(i) Supp Bk is an effective \mathbb{Q} -divisor with simple normal crossing support and proper fractional coefficients. Since \mathcal{P} is nef and big ($\kappa(\mathcal{P}) = \kappa(K + D) = 2$), the Kawamata-Viehweg vanishing theorem (see for example [Laz04, 9.1.18]) says that $h^1(2K + D) = 0$ for $i > 0$. This completes the proof. \square

Remark. Note that every Q_i contains some maximal twig T_j of D with $d(T_j) \geq 3$. Indeed, it is clear if $Q_i = [(2)_m, 3, 1, 2]$ for some m and the general case follows by induction on the number of components of Q_i .

Corollary 3.3. *Assume $\bar{E} \subseteq \mathbb{P}^2$ is not rectifiable. Then $t \leq 10$ in case $c \geq 4$ and $t \leq 9$ in case $c \leq 3$. In particular, \bar{E} has at most five cusps.*

Proof. The remark above gives $\delta(D) \leq \frac{1}{3}c + \frac{1}{2}(t - c) = \frac{1}{2}t - \frac{1}{6}c$. By 2.4(i) $\kappa(K + E) \geq 0$, so (\star) gives $t \leq \frac{2}{3}c + \epsilon(c) + 8$. For $c \leq 2$ we get $t \leq 9$. Assume $c \geq 3$. By 2.6(ii) $\kappa(K + D) = 2$. Since $2c \leq t$, by (\star) and (\diamond)

$$\frac{3}{4}t \leq t - \frac{1}{2}c \leq 4 + \delta(D) \leq 4 + e(D) \leq 8,$$

so $t \leq \frac{32}{3} < 11$. If $c = 3$ then we have in fact $t - \frac{3}{2} \leq 8$, so $t \leq 9$. \square

Remark. We note that if $\bar{E} \subseteq \mathbb{P}^2$ is rectifiable then it has at most eight cusps by [Ton05].

4. FIVE CUSPS

In this section we assume that $\bar{E} \subseteq \mathbb{P}^2$ is a rational cuspidal curve which is non-rectifiable. In particular $h^0(2K + D) \geq h^0(2K + E) > 0$. By 3.3 $t \leq 10$, so \bar{E} has at most five cusps. Therefore, to prove the theorem 1.1 we can assume that $t = 10$ and $c = 5$. For an (ordered) rational chain with negative definite intersection matrix we put $u(T) = e(T) - \delta(T) \geq 0$.

Lemma 4.1. *Assume $T = T' + C + T''$ is a rational chain with a negative definite intersection matrix, with a unique (-1) -curve C and having $d(T) = 1$. Assume T', T'' are nonempty and $d(T') \leq d(T'')$. Put $\bar{u}(T) = u(T') + u(T'')$ (we assume that the tips of T are the first components of T' and T''). Then:*

- (i) $d(T')$ and $d(T'')$ are coprime,
- (ii) $\bar{u}(T) \geq 0$ and the equality holds only if $T = [2, 1, 3]$,
- (iii) if $T' = [2]$ then $T'' = [(2)_k, 3]$ for some $k \geq 0$ and $\bar{u}(T) = 1 - \frac{3}{2k+3}$,
- (iv) if $T' = [2, 2]$ then $T'' = [(2)_k, 4]$ for some $k \geq 0$ and $\bar{u}(T) = \frac{4}{3} - \frac{4}{3k+4}$,
- (v) if $T' = [3]$ then $T'' = [(2)_k, 3, 2]$ for some $k \geq 0$ and $\bar{u}(T) = 1 - \frac{4}{3k+5}$,
- (vi) if $T' = [4]$ then $T'' = [(2)_k, 3, 2, 2]$ for some $k \geq 0$ and $\bar{u}(T) = 1 - \frac{5}{4k+7}$.

Proof. Let A and A' be the components of T' and T'' meeting C respectively. By 2.1 $d(T')d(C+T'') - d(T'-A')d(T'') = d(T) = 1$, so $d(T')$ and $d(T'')$ are coprime. If $\bar{u}(T) = 0$ then T' and T'' are irreducible, so $d([- (T')^2, 1, - (T'')^2]) = 1$, which happens only if $T = [2, 1, 3]$. For (iii) note that since T contracts to a (-1) -curve, we necessarily have $T = [2, 1, 3, (2)_k]$ for some $k \geq 0$. We compute $d([(2)_k, 3]) = 2k + 3$, which gives the result. The remaining cases are done analogously. \square

Corollary 4.2. *Let T be as above. Put $\bar{\delta}(T) = \delta(T') + \delta(T'')$. If $d(T') \leq 4$ and $0 < \bar{u}(T) < \frac{1}{2}$ then:*

- (i) $T = [2, 1, 3, 2]$ and $(\bar{u}(T), \bar{\delta}(T)) = (\frac{2}{5}, \frac{7}{10})$ or
- (ii) $T = [3, 1, 2, 3]$ and $(\bar{u}(T), \bar{\delta}(T)) = (\frac{1}{5}, \frac{8}{15})$ or
- (iii) $T = [2, 2, 1, 4]$ and $(\bar{u}(T), \bar{\delta}(T)) = (\frac{1}{3}, \frac{7}{12})$ or
- (iv) $T = [4, 1, 2, 2, 3]$ and $(\bar{u}(T), \bar{\delta}(T)) = (\frac{2}{7}, \frac{11}{28})$.

Proof. If $d(T') \leq 4$ then $T' = [2], [3], [4], [2, 2]$ or $[2, 2, 2]$. In the last case $\bar{u}(T) \geq u([2, 2, 2]) = \frac{1}{2}$. For the remaining cases use 4.1. \square

Put $\bar{u}_i = \bar{u}(Q_i)$ and $\bar{\delta}_i = \bar{\delta}(Q_i)$. We have $\delta(D) \neq e(D)$, otherwise by 4.1(ii) $e(D) = 5(\frac{1}{2} + \frac{1}{3}) > 4$, which contradicts (\diamond) . The inequalities (\star) and (\diamond) give

$$(4.1) \quad \frac{7}{2} \leq \delta(D) < e(D) \leq 4.$$

In particular

$$0 < \sum_{i=1}^5 \bar{u}_i \leq \frac{1}{2}.$$

By renumbering twigs we can assume that $Q_i - C_i = T_i + T_{i+5}$ and $d(T_i) < d(T_{i+5})$.

Proposition 4.3. *With the notation as above $\bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0$ and:*

- (i) *either $Q_4 = [3, 1, 2, 3]$ and $Q_5 = [3, 1, 2, 3]$ or*
- (ii) *$\bar{u}_4 = 0$ and $Q_5 = [n, 1, (2)_{n-2}, 3]$ for some $5 \leq n \leq 9$ or*
- (iii) *$\bar{u}_4 = 0$ and $Q_5 = [5, 2, 1, 3, 2, 2, 3]$.*

Proof. By renumbering twigs we may assume $\bar{u}_1, \bar{u}_2, \bar{u}_3 \leq \bar{u}_4, \bar{u}_5$, $d(T_1) \leq d(T_2) \leq d(T_3)$ and $\bar{\delta}_4 \geq \bar{\delta}_5$. Suppose $\bar{u}_4, \bar{u}_5 \neq 0$. We have $\bar{\delta}_4 + \bar{\delta}_5 \geq \frac{7}{2} - 3(\frac{1}{2} + \frac{1}{3}) = 1$. Then $\bar{\delta}_4 \geq \frac{1}{2}$, so $d(T_4) \leq 3$.

Suppose $d(T_4) = 2$. By 4.2 $Q_4 = [2, 1, 3, 2]$, so $\frac{2}{d(T_5)} \geq \bar{\delta}_5 \geq 1 - \frac{7}{10} = \frac{3}{10}$ and $\bar{u}_5 \leq \frac{1}{2} - \frac{1}{5} = \frac{1}{10}$. Thus $d(T_5) \leq 6$ and $\frac{1}{d(T_{10})} \geq \frac{3}{10} - \frac{1}{d(T_5)}$. By 4.2 we have $d(T_5) \in \{5, 6\}$. Since $u([3, 2]), u([2, 3]), u([(2)_4]), u([(2)_5]) \geq \frac{1}{5}$, by 2.2 we see that $T_5 = [5]$ or $T_5 = [6]$. If $T_5 = [6]$ then we simultaneously have $d(T_{10}) \leq 7$ and $T_{10} = [(2)_k, 3, (2)_4]$ for some $k \geq 0$, which is impossible. Thus $T_5 = [5]$ and then $d(T_{10}) \leq 10$. This is possible only for $T_{10} = [2, 2, 2]$ or $T_{10} = [3, 2, 2, 2]$. Then $\bar{u}_5 = \frac{1}{2}$ and $\bar{u}_5 = \frac{1}{3}$ respectively, a contradiction.

Suppose $d(T_4) = 3$. If $T_4 = [2, 2]$ then by 4.2 $\bar{\delta}_5 \geq 1 - \frac{7}{12} = \frac{5}{12}$ (in particular $d(T_5) \leq 4$) and $\bar{u}_5 \leq \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, which contradicts 4.2. Thus $T_4 = [3]$ and $Q_4 = [3, 1, 2, 3]$. We obtain $\bar{\delta}_5 \geq \frac{7}{15}$ and $\bar{u}_5 \leq \frac{3}{10}$, so $Q_5 = [3, 1, 2, 3]$ by 4.2. By (4.1) $\bar{\delta}_1 + \bar{\delta}_2 + \bar{\delta}_3 \geq \frac{7}{2} - 2 \cdot \frac{8}{15} = \frac{73}{30}$, so $\bar{\delta}_3 \geq \frac{73}{30} - 2 \cdot \frac{5}{6} = \frac{23}{30}$, hence $d(T_1) = d(T_2) = d(T_3) = 2$. On the other hand $\bar{u}_1 + \bar{u}_2 + \bar{u}_3 \leq \frac{1}{2} - \bar{u}_4 - \bar{u}_5 = \frac{1}{10}$, so by 4.2 $\bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0$. This is the case (i).

We may now assume $\bar{u}_i = 0$ for $i \neq 5$. Then (4.1) gives

$$(4.2) \quad \frac{1}{6} \leq \bar{\delta}_5 \leq \bar{e}_5 \leq \frac{2}{3},$$

where $\bar{e}_5 = e(T_5) + e(T_{10})$. The lower bound $\bar{\delta}_5 \geq \frac{1}{6}$ gives $d(T_5) \leq 11$. Since $d(T_5)$ and $d(T_{10})$ are coprime, we have in fact $\bar{\delta}_5 > \frac{1}{6}$ and hence $\bar{u}_5 < \frac{1}{2}$. In case $Q_5 = [2, 1, 3]$ and in all cases listed in 4.2 the inequality $\bar{e}_5 = \bar{\delta}_5 + \bar{u}_5 \leq \frac{2}{3}$ fails, hence $d(T_5) \geq 5$. It follows in particular that the twig T_5 , cannot consists only of (-2) -curves, otherwise $\frac{2}{3} \geq e(T_5) = 1 - \frac{1}{d(T_5)} \geq \frac{4}{5}$.

Suppose $T_5 = [n]$ for some $5 \leq n \leq 11$. Then $T_{10} = [(2)_m, 3, (2)_{n-2}]$ for some $m \geq 0$, so $d(T_{10}) = (m+2)n - 1$. The inequality $\bar{u}_5 < \frac{1}{2}$ is equivalent to $mn \leq 2$, hence $m = 0$. For $n = 10, 11$ the inequality $\bar{\delta}_5 > \frac{1}{6}$ fails, hence $n \leq 9$. This is the case (ii).

Since $d(T_5) \leq 11$, it is easy to list the remaining possibilities for T_5 with $e(T_5) < \frac{2}{3}$ (note that $e(T_5) \neq \frac{2}{3}$, as $e(T_{10}) \neq 0$). These are: $[2, k]$ and $[k, 2]$ for $3 \leq k \leq 6$, $[3, 3]$, $[3, 4]$, $[4, 3]$, $[2, 3, 2]$, $[3, 2, 2]$, $[4, 2, 2]$, $[2, 3, 2, 2]$, $[3, 2, 2, 2]$ and $[3, 2, 2, 2, 2]$. For each such T_5 we computed explicitly T_{10} for which $\bar{u}_5 < \frac{1}{2}$ and we checked that in cases other than $T_5 = [5, 2]$, $T_{10} = [3, 2, 2, 3]$ one of the inequalities $\bar{\delta}_5 > \frac{1}{6}$ or $\bar{e}_5 \leq \frac{2}{3}$ fails. \square

Decompose $\pi: X \rightarrow \mathbb{P}^2$ into blow-ups $\sigma_1 \circ \dots \circ \sigma_s$. Let m_i , $i = 1, \dots, s$ be the multiplicity if the center of σ_i as a point on the respective proper transform of \bar{E} . The non-increasing sequence of multiplicities of a given cusp q_i and of all centers lying above it is the *multiplicity sequence of the cusp* q_i . Since $p_a(E) = 0$, the genus formula reads as

$$\sum_{i=1}^s \binom{m_i}{2} = \binom{\deg \bar{E} - 1}{2}.$$

The multiplicity sequence for a cusp with $Q_i = [2, 1, 3]$ is $(2, 1, 1)$. For $Q_i = [3, 1, 2, 3]$ it is $(3, 2, 1, 1)$, for $Q_i = [n, 1, (2)_{n-2}, 3]$ it is $(n, n-1, (1)_{n-1})$ and for $Q_i = [5, 2, 1, 3, 2, 2, 3]$ it is $(9, 7, (2)_3, (1)_2)$. Thus $\binom{\deg \bar{E}-1}{2} = 11$ in case (i), $\binom{\deg \bar{E}-1}{2} = n^2 - 2n + 5$ in case (ii) and $\binom{\deg \bar{E}-1}{2} = 64$ in case (iii). These equations have no solutions in natural numbers, which completes the proof of the theorem.

5. FOUR CUSPS

Let $\bar{E} \subseteq \mathbb{P}^2$ be a rational cuspidal curve with cusps q_1, \dots, q_c . For $i = 1, \dots, c$ let $\bar{m}_i = (m_{i,0}, m_{i,1}, \dots, m_{i,k_i})$ be the multiplicity sequence of q_i as defined above. Note that we do not omit 1's from the sequence. The resolution $\pi: X \rightarrow \mathbb{P}^2$ can be described in terms of Hamburger-Noether pairs (characteristic pairs). For a cusp q_i we denote the sequence of H-N pairs by

$$\left(\begin{smallmatrix} c_{i,1} \\ p_{i,1} \end{smallmatrix} \right), \dots, \left(\begin{smallmatrix} c_{i,h_i} \\ p_{i,h_i} \end{smallmatrix} \right),$$

where $\gcd(c_{i,j}, p_{i,j}) = c_{i,j+1}$ for $j < h_i$ and $\gcd(c_{i,h_i}, p_{i,h_i}) = 1$. In general the pairs depend on a choice of the system of local parameters $\{x_1, y_1\}$ around q_i . We choose our parameters so that $L_1 = \{x_1 = 0\}$ is the tangent direction of \bar{E} . Recall that the first H-N pair is defined as

$$c_{i,1} = (\bar{E} \cdot \{x_1 = 0\})_{q_i}, \quad p_{i,1} = (\bar{E} \cdot \{y_1 = 0\})_{q_i},$$

where $(\)_{q_i}$ denotes the local intersection index at q_i . The inductive step defining $\{x_2, y_2\}$, and hence the remaining part of the sequence of H-N pairs for q_i , is as follows. We blow up over q_i until the proper transform E' of \bar{E} meets the inverse image not in a node. Denote the last produced exceptional curve by L_2 and the point of intersection with E' by \tilde{q}_i . If E' is smooth at \tilde{q}_i we put $h_i = 1$. Otherwise we choose local parameters $\{x_2, y_2\}$ around \tilde{q}_i , so that $\{x_2 = 0\} = L_2$ and $p_{i,2} = (E' \cdot \{y_2 = 0\})_{\tilde{q}_i}$ is the multiplicity of $\tilde{q}_i \in E'$. This forces $p_{i,j} \leq c_{i,j}$. It follows from the definition that $p_{i,1} = m_{i,0}$.

A more complete reference to H-N pairs is [Rus80]. Here we basically followed [CNKR09, 1.12]. By $\binom{c}{p}_k$ we mean a sequence of pairs $\binom{c}{p}, \dots, \binom{c}{p}$ of length k . For a short proof of the following result see for example [PK10, 2.10].

Lemma 5.1. *With the notation as above:*

$$\begin{aligned} (i) \quad & \sum_{j=1}^{k_i} m_{i,j} = c_{i,1} + \sum_{j=1}^{h_i} p_{i,j} - 1, \\ (ii) \quad & \sum_{j=1}^{k_i} m_{i,j}^2 = \sum_{j=1}^{h_i} c_{i,j} p_{i,j}. \end{aligned}$$

Put $M(q_i) = c_{i,1} + \sum_{j=1}^{h_i} p_{i,j} - 1$ and $I(q_i) = \sum_{j=1}^{h_i} c_{i,j} p_{i,j}$.

Example 5.2. If $Q_i = [2, 1, 3]$ then $h_i = 1$, $\binom{c_{i,1}}{p_{i,1}} = \binom{3}{2}$, $M(q_i) = 4$ and $I(q_i) = 6$. If $Q_i = [5, 2, 1, 3, 2, 2, 3]$ then $h_i = 1$, $\binom{c_{i,1}}{p_{i,1}} = \binom{16}{9}$, $M(q_i) = 24$ and $I(q_i) = 144$.

Corollary 5.3. *Let $\gamma = -E^2$ and let $d = \deg \bar{E}$. Then*

$$(i) \quad \gamma - 2 + 3d = \sum_i M(q_i),$$

$$(ii) \quad \gamma + d^2 = \sum_i I(q_i).$$

Proof. Let $C \subseteq X$ be an irreducible curve on a smooth projective surface. Let $p \in C$ be a singular point of C having multiplicity m and let $\sigma: X' \rightarrow X$ be a blow-up at p . Denote the exceptional curve by L and the proper transform of C on X' by C' . Then $K_X \cdot C = \sigma^* K_X \cdot C' = (K_{X'} - L) \cdot C' = K'_{X'} \cdot C' - m$. Also $C^2 = \sigma^* C \cdot C' = (C' + mL) \cdot C' = C'^2 + m^2$. It follows that the sum of all multiplicities $m_{i,j}$ equals $K_X \cdot E - K_{\mathbb{P}^2} \cdot \bar{E} = \gamma - 2 + 3d$ and the sum of their squares equals $\bar{E}^2 - E^2 = d^2 + \gamma$. \square

Proposition 5.4. *If $\bar{E} \subseteq \mathbb{P}^2$ is non-rectifiable and $c = 4$ then $t \leq 9$.*

Proof. Suppose $t \geq 10$. By 3.3 $t = 10$. Since $\kappa(2K + E) > 0$ and $\kappa(K + 2) = 2$, (\star) and (\diamond) give

$$\delta(D) = e(D) = 4,$$

so all maximal twigs of D are tips. In fact (\diamond) gives also $\mathcal{P}^2 = 3$ and $K \cdot (K + D) = 1$. By 4.1(ii), renumbering the twigs if necessary, we may assume that T_1, T_2, T_3, T_4 are equal $[2]$ and T_5, T_6, T_7, T_8 are equal $[3]$ and that $T_i, T_{i+4} \subseteq Q_i$ for $i \leq 4$. Let $T_9 \subseteq Q_1$ and $T_{10} \subseteq Q_1 \cup Q_2$ be the remaining two maximal twigs of D . We may assume that in case $T_{10} \subseteq Q_2$ we have $d(T_9) \leq d(T_{10})$ and in case $T_{10} \subseteq Q_1$ the tip T_9 is created before T_{10} in the process of resolving the singularity $q_1 \in \bar{E}$. We have $\delta(T_9) + \delta(T_{10}) = \frac{2}{3}$, which gives $(T_9, T_{10}) = ([2], [6])$ or $(T_9, T_{10}) = ([6], [2])$ or $(T_9, T_{10}) = ([3], [3])$. The Noether formula for X reads as $K^2 + \#D = 10$. We have $1 = K \cdot (K + D) = K \cdot (K + D - E) + \gamma - 2$, and $K \cdot Q_i = 0$, $\#Q_i = 3$ for $i = 3, 4$, so we obtain

$$\#Q_1 + \#Q_2 = K \cdot Q_1 + K \cdot Q_2 + \gamma.$$

We define a subsequence of the sequence of characteristic pairs to be of type $*(n, k)$ if it is equal $\binom{\alpha n}{\alpha n}_k, \binom{\alpha n}{\alpha n - \alpha}$ for some $n \geq 2$, $\alpha > 0$ and $k \geq 0$. Let $p \in C$ be a point on a smooth curve and let \tilde{C} and C' be the total and proper transforms of C after performing blowups over p according to a sequence of type $*(n, k)$. Then

$$\tilde{C} = [-C'^2 + 1, (2)_{k-1}, 3, (2)_{n-2}, 1, n]$$

if $k \neq 0$ and $\tilde{C} = [-C'^2 + 2, (2)_{n-2}, 1, n]$ otherwise. In particular this produces a $(-n)$ -tip. We have $K \cdot \tilde{C} - K \cdot C' = n - 1$ and $\#\tilde{C} - 1 = k + n$. The subsequences producing the tips T_9 and T_{10} are of type $*(d(T_9), k)$ and $*(d(T_{10}), l)$ for some $k, l \geq 0$. Therefore

$\#Q_1 + \#Q_2 = 6 + k + l + d(T_9) + d(T_{10})$ and $K \cdot Q_1 + K \cdot Q_2 = d(T_9) + d(T_{10}) - 2$. The Noether formula reduces now to

$$(5.1) \quad l = \gamma - k - 8.$$

We have the following five possibilities for sequences of H-N pairs.

Case 1. $T_9 = [2]$, $T_{10} = [6]$, $T_{10} \subseteq Q_1$.

The sequence of pairs for q_2 is $\binom{3}{2}$ and for q_1 it is $\binom{36}{24}$, $\binom{12}{12}_k$, $\binom{12}{6}$, $\binom{6}{6}_l$, $\binom{6}{5}$. Then $M(q_1) = 70 + 12k + 6l$, $I(q_1) = 966 + 144k + 36l$, $M(q_2) = 4$, $I(q_2) = 6$. Using equations 5.3 and (5.1) we eliminate γ and l and we obtain the relation $d^2 - 21d = 444 + 66k$. Then $(2d + 1)^2 = 6 \pmod{11}$, which is impossible.

Case 2. $T_9 = [6]$, $T_{10} = [2]$, $T_{10} \subseteq Q_1$.

The sequence of pairs for q_2 is as above and for q_1 it is $\binom{36}{24}$, $\binom{12}{12}_k$, $\binom{12}{10}$, $\binom{2}{2}_l$, $\binom{2}{1}$. Then $M(q_1) = 70 + 12k + 2l$ and $I(q_1) = 986 + 144k + 4l$. As above we get $d^2 - 9d = 768 + 110k$, which gives $(2d + 1)^2 = 3 \pmod{5}$, a contradiction.

Case 3. $T_9 = [3]$, $T_{10} = [3]$, $T_{10} \subseteq Q_1$.

The sequence of pairs for q_2 is as above and for q_1 it is $\binom{27}{18}$, $\binom{9}{9}_k$, $\binom{9}{6}$, $\binom{3}{3}_l$, $\binom{3}{2}$. Then $M(q_1) = 52 + 9k + 3l$ and $I(q_1) = 546 + 81k + 9l$. We get $d^2 - 12d = 324 + 48k$, which gives $(d - 6)^2 = 24 \pmod{48}$, a contradiction.

Case 4. $T_9 = [3]$, $T_{10} = [3]$, $T_{10} \subseteq Q_2$.

The sequences for q_1 and q_2 are $\binom{9}{6}$, $\binom{3}{3}_k$, $\binom{3}{2}$ and $\binom{9}{6}$, $\binom{3}{3}_l$, $\binom{3}{2}$ respectively. Then $M(q_1) = 16 + 3k$, $I(q_1) = 60 + 9k$, $M(q_2) = 16 + 3l$ and $I(q_2) = 60 + 9l$. We get $d^2 - 12d + 12 = 0$, a contradiction.

Case 5. $T_9 = [2]$, $T_{10} = [6]$, $T_{10} \subseteq Q_2$.

The sequences for q_1 and q_2 are $\binom{6}{4}$, $\binom{2}{2}_k$, $\binom{2}{1}$ and $\binom{18}{12}$, $\binom{6}{6}_l$, $\binom{6}{5}$ respectively. Then $M(q_1) = 10 + 2k$, $I(q_1) = 26 + 4k$, $M(q_2) = 34 + 6l$ and $I(q_2) = 246 + 36l$. We now eliminate k and l and we get $d^2 - 24d + 52 = -5\gamma$, hence $(2d + 1)^2 = 3 \pmod{5}$, a contradiction. \square

By 1.1, 3.3 and 5.4 we obtain the following result.

Corollary 5.5. *Let $\bar{E} \subseteq \mathbb{P}^2$ be a rational cuspidal curve defined over complex numbers. If $\bar{E} \subseteq \mathbb{P}^2$ is not rectifiable then the tree of the exceptional divisor for its minimal embedded resolution has at most nine maximal twigs.*

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